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QUALITATIVE ANALYSIS OF MOTION OF A HEAVY SOLID BODY ON A SMOOTH HORIZONTAL PLANE*

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The motion of a solid body on a stationary absolutely smooth horizontal surface in a gravitational field is considered. The surface that bounds the body is convex, and the body differs little from a dynamically and geometrically symmetric one. This difference is defined by the magnitude if the small parameters ε. The unperturbed problem (when $\varepsilon=0$) is integrable /1/. The basic aim is the investigation of motion for $0<\varepsilon\ll 1$. The nondegeneracy of Hamiltonian function of the unperturbed motion is shown and on the basis of Kolmogorov theorem /2,3/ is established that the perpetual closeness of variables "action" to their initial values which correspond to conditional periodic motions in the unperturbed problem. By this is established the smallness of variation of basic geometrical characteristics of the unperturbed motion, when the solid body differs little from the geometrically and dynamically symmetric one.

Considerable results have been achieved to the present in solving the problem of existence and stability of steady motions of solid bodies and gyrostates on a stationary plane, in particular, absolutely smooth one. The steady motions are not necessarily rotation of the body about the vertical. Besides, the stability was considered in a strictly nonlinear formulation. The basic results here were obtained in papers /4-6/. The qualitative analysis of motion of a heavy homogeneous triaxial ellipsoid is carried out on a smooth horizontal plane /7/ on the assumption of its closeness to the sphere.

1. Let $OXYZ$ be a fixed coordinate system with its origin at point O of horizontal plane OXY on which the body is moving, the OZ axis directed vertically upward, and $Gxyz$ the coordinate system rigidly attached to the body. The origin of the attached coordinate system is at the center of mass of the body and its axes are directed along its principal central axes of inertia. The mutual orientation of the attached and the fixed system of coordinates is specified by the Euler angles ψ, θ, φ.

The considered mechanical system is holonomic and has five degrees of freedom. As the generalized coordinates we take the three Euler angles and two coordinates X_G and Y_G of the center of mass in the system of coordinates $OXYZ$. The third coordinate Z_G — the distance of the center of mass of the body — is a function of angles θ and φ, for a given form of surface bounding the body.

Let m be the mass of the body, g the acceleration of free fall, and A, B and C the moments of inertia relative to axes Gx, Gy, and Gz, respectively. The kinetic and potential energies of the body are as follows:

$$\begin{aligned} &T = {}^1/_2 m\,(X_G^{\cdot 2} + Y_G^{\cdot 2} + Z_G^{\cdot 2}) + {}^1/_2\,(Ap^2 + Bq^2 + Cr^2) \\ &\Pi = mgZ_G\,(\theta, \varphi) \\ &p = \psi^{\cdot} \sin\theta \sin\varphi + \theta^{\cdot} \cos\varphi,\quad q = \psi^{\cdot} \sin\theta \cos\varphi - \theta^{\cdot} \sin\varphi \\ &r = \psi^{\cdot} \cos\theta + \varphi^{\cdot} \end{aligned} \tag{1.1}$$

The generalized coordinates ψ, X_G, Y_G are cyclic (ignorable). Hence the projection of p_ψ on the vertical vector of the body kinetic moment relative to point G is constant, as is also constant the velocity of projection Q of the center of mass on plane OXY (Fig.1).

The presence of three cyclic coordinates enable us to reduce the problem of the solid body motion the investigation of the system with two degrees of freedom. Without loss of generality, we assume that the velocity of point Q is zero. At the same time we consider p_φ in the Hamilton functions $H = H\,(\theta, \varphi, p_\theta, p_\varphi, p_\psi)$ as parameters.

Let the bounding surface of the body differ little from a surface of revolution with the Gz axis and the body be close to a dynamically symmetric one. It is then possible to set $Z_G = f(\theta) + \varepsilon f_1(\theta, \varphi)$, $B = A(1+\varepsilon)$ $(0 \leqslant \varepsilon \ll 1)$, and the Hamiltonian reduced system is written in the form

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$$H = H_0(\theta, p_\theta, p_\varphi, p_\psi) + \varepsilon H_1(\theta, \varphi, p_\theta, p_\varphi, p_\psi, \varepsilon)$$

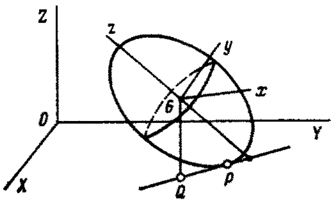


Fig.1

The unperturbed motion (when $\varepsilon = 0$) is the motion of a dynamically and geometrically symmetric body, for instance, a solid homogeneous body of revolution. The Hamiltonian H_0 corresponding to the unperturbed motion is of the form

$$H_0 = \frac{p_\theta^2}{2(A + m\rho^2)} + \frac{(p_\psi - p_\varphi \cos\theta)^2}{2A\sin^2\theta} + mgf(\theta) + \frac{p_\varphi^2}{2C} \tag{1.2}$$

where $\rho = \rho(\theta) = \pm f'(\theta)$ is the distance from the tangency point P of the body with the plane OXY to point Q. When $\varepsilon = 0$ the generalized velocities and the moments are linked by the relations

$$\psi^\cdot = \frac{p_\psi - p_\varphi\cos\theta}{A\sin^2\theta}, \quad \varphi^\cdot = \left(\frac{1}{C} - \frac{1}{A}\right)p_\varphi + \frac{p_\varphi - p_\psi\cos\theta}{A\sin^2\theta}, \quad \theta^\cdot = \frac{p_\theta}{A + m\rho^2} \tag{1.3}$$

2. Let us consider the unperturbed motion in more detail. Formula (1.2) implies that in unperturbed motion one more coordinate, the angle φ, is cyclic. The corresponding momentum p_φ, projection of the momentum of the body on the axis of symmetry, is constant, and the investigation of the unperturbed motion reduces to the consideration of the reduced system with only one degree of freedom. The kinetic and potential energies of the reduced system are defined by the formulas

$$T_* = {}^1/_2(A + m\rho^2)\theta^{\cdot 2}, \quad \Pi_* = \frac{(p_\psi - p_\varphi\cos\theta)^2}{2A\sin^2\theta} + mgf(\theta) + \frac{p_\varphi^2}{2C} \tag{2.1}$$

The variation of angle $\theta = \theta(t)$ is obtained using the integral of energy $T_* + \Pi_* = h =$ const. Denoting by θ_0 the initial value of angle θ, we obtain

$$\pm\int_{\theta_0}^{\theta}\sqrt{\frac{A + m\rho^2}{2(h - \Pi_*)}}\,d\theta = t \tag{2.2}$$

When $\theta = \theta(t)$ is known, the variation of angles $\psi = \psi(t), \varphi = \varphi(t)$ is obtained from (1.3) by quadratures.

Let us first consider such motion of the body when its axis of rotation is not vertical, i.e. the angle θ during the whole time of motion cannot be equal 0 or π. For this it is sufficient to stipulate $p_\psi \neq \pm p_\varphi$. It is obvious that $h - \Pi_*(\theta_0) > 0$, and for $p_\psi \neq \pm p_\varphi$ the quantity $h - \Pi_*(\theta)$ becomes negative, if $\theta \to 0$ or π. Hence angle θ is included between two real roots of equation $h - \Pi_*(\theta) = 0$ lying between 0 and π. If θ_1, θ_2 are two different $(\theta_2 > \theta_1)$ simple roots of that equation and in the interval between these roots $h > \Pi_*(\theta)$, then angle θ fluctuates between θ_1 and θ_2 in conformity with (2.2). The period of these oscillations is

$$\tau = \int_{\theta_1}^{\theta_2}\sqrt{\frac{2(A + m\rho^2)}{h - \Pi_*(\theta)}}\,d\theta \tag{2.3}$$

When throughout the time of motion $\theta = \theta_0$ we have a regular precession of a solid body. Its center of mass is stationary, and the angular velocities $\psi^\cdot, \varphi^\cdot$ of rotation of the body around the vertical and the axis of symmetry are constant. The point P of contact of the body with the OXY plane describes on the latter a circle with its center at point Q, and on the body surface, a circle whose plane is perpendicular to the axis of symmetry of the body. In the particular case, when $\varphi^\cdot = 0$ the body touches the plane with one point of its surface.

If h is fixed, the regular precession is possible only then when θ_0 is a multiple root of equation $h - \Pi(\theta) = 0$, i.e. when $\theta = \theta_0$ satisfies the system

$$\Pi_*(\theta) = h, \ \Pi_*'(\theta) = 0 \tag{2.4}$$

For any function $f(\theta)$ it is possible to obtain regular precession with arbitrarily specified angle θ_0 of nutation by suitable selection of variables p_ψ, p_φ, h.

Indeed the second of Eqs.(2.4) can be written as follows:

$$2p_\varphi = ap_\psi \pm \sqrt{(a^2 - 4)p_\psi^2 + 4b}, \quad a = (1 + \cos^2\theta)/\cos\theta \tag{2.5}$$
$$b = Amgf'\sin^3\theta/\cos\theta$$

Since $|a| > 2$, by the suitable selection of p_ψ the radicand of (2.5) can be made positive and by that satisfy the second of Eqs.(2.4). While the first of Eqs.(2.4) is satisfied if we set $h = \Pi_*(\theta_0)$.

As an example, Fig.2 represent the behavior of trajectories of the reduced system in the case of $p_\psi \neq \pm p_\varphi$, when $\Pi_*(\theta)$ has one local minimum (Fig.2,a) and when $\Pi_*(\theta)$ has two local minima and one local maximum (Fig.2,b).

Let us briefly consider such motion of the body when its axis of symmetry may pass through the vertical position. When the axis of symmetry may pass the position $\theta = 0$, then necessarily must be satisfied the equality $p_\psi = p_\varphi$. Opening the indefiniteness in the expression for function $\Pi_*(\theta)$, we obtain

$$\Pi_* = \frac{p_\varphi^2}{2A}\operatorname{tg}^2\frac{\theta}{2} + mgf(\theta) + \frac{p_\varphi^2}{2C} \tag{2.6}$$

Since $f'(0) = 0$, hence also $\Pi_*'(0) = 0$. This means that there exists steady motion of the body, which is a rotation about the vertically located axis of symmetry at constant angular velocity r_0. Analyzing the character of the extremum of function (2.6) at point $\theta = 0$, and taking into account $p_\varphi = Cr_0$, we obtain that the sufficient condition of stability of such motion with respect to θ and $\theta^{\cdot}$ is the fulfillment of the inequality

$$C^2 r_0^2 + 4Amgf''(0) > 0 \tag{2.7}$$

When the sign of this inequality is reversed, we have instability. These conditions conform to the respective results of /4,5/.

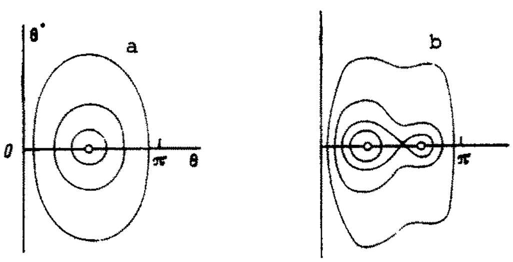
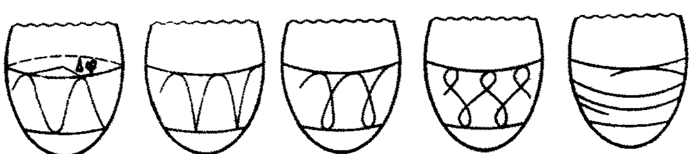
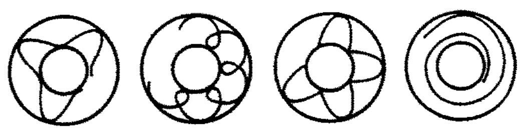


Fig.2



a b c d e

Fig.3



a b c d

Fig.4

If the axis of symmetry can pass through the position when $\theta = \pi$ (a reversal), then $p_\psi = -p_\varphi$. The reduced potential energy Π_* is of the form of (2.6), where in the first term it is necessary to substitute $\operatorname{ctg}\theta/2$ for $\operatorname{tg}\theta/2$. The condition of stability of rotation about the vertical is written in the form of inequality (2.7) in which the derivative f'' must be calculated for $\theta = \pi$.

If $p_\psi = p_\varphi = 0$, then the body axis of symmetry can pass during its motion through both singular positions $\theta = 0$ and $\theta = \pi$. When $p_\psi = p_\varphi = 0$, from (1.3) we obtain $\psi = \text{const}$, $\varphi = \text{const}$, i.e. the body moves so that its axis of symmetry is in the fixed vertical plane all the time. The dependence of the angle of nutation on time is obtained from (2.2), where it is necessary to set $\Pi_* = mgf(\theta)$.

Consider the properties of traces of the touching point on the plane and on the body surface that are not regular precession. If angle θ fluctuates between θ_1 and θ_2 with period τ, then ψ and φ obtain during time τ some continuous increments. In that case the trace of contact point on the body surface is contained between two parallel lines, and on the plane between two concentric circles, what is demonstrated in Figs.3 and 4. In Fig.3,a a case is represented when $\varphi^{\cdot}$ does not change its sign; in Fig.3,b $\varphi^{\cdot}$ vanishes at $\theta = \theta_1$; Fig.3,c and d correspond to such motions of the body, when in the period τ of one of its oscillation with respect to the angle θ, $\varphi^{\cdot}$ changes its sign, respectively once and twice; Fig.3,e corresponds to the separatrix in plane $\theta, \theta^{\cdot}$. In Fig.4,a, b and c the quantity $\psi^{\cdot}$ does not change its sign in time τ changes its sign once, and vanishes for $\theta = \theta_1$; Fig.4,d corresponds to the separatrix.

3. Subsequently we shall assume that $p_\psi \neq \pm p_\varphi$, i.e. that the existence of such body motions when its axis of symmetry could pass through the singular positions $\theta = 0, \pi$, are excluded. Isoenergetic curves $H_0 = \text{const}$ in the plane θ, p_θ in principle do not differ from respective curves $T_* + \Pi_* = h$ in plane $\theta, \theta^\cdot$. Having eliminated in the consideration asymptotic motions corresponding to regular precessions of the body, we obtain that the isoenergetic curves in plane θ, p_θ are closed and on them $0 < \theta_1 \leqslant \theta \leqslant \theta_2 < \pi$.

For the investigation of the perturbed motion $(0 < \varepsilon \ll 1)$ it is convenient to introduce the canonical variables action, the angle $\theta, \varphi, p_\theta, p_\varphi \to w_1, w_2, I_1, I_2$. The variables action are specified by the equalities

$$I_1(I_2, H_0) = \frac{1}{2\pi}\oint\left\{(A + m\rho^2)\left[2H_0 - \frac{I_2^2}{C} - \frac{(p_\psi - I_2\cos\theta)^2}{A\sin^2\theta} - 2mgf(\theta)\right]\right\}^{1/2} d\theta, \quad I_2 = p_\varphi \tag{3.1}$$

Integration is carried out over closed curves $H_0 = \text{const}$.

In variables w_1, w_2, I_1, I_2 the Hamiltonian of perturbed motion assumes the form

$$H = H_0(I_1, I_2) + \varepsilon H_1(I_1, I_2, w_1, w_2) \tag{3.2}$$

The perturbation εH_1 is of period 2π with respect to variables w_1 and w_2. The dependence on parameters of the problem, including on p_ψ, is not indicated in (3.2).

The frequencies of unperturbed motions $w_i(I_1, I_2) = \partial H_0/\partial I_i$ $(i = 1, 2)$ are analytic functions of their arguments. The variables I_1, I_2 when $\varepsilon = 0$ are constants and equal to their initial values.

Let us consider the isoenergetic level $H_0 = h = \text{const}$. On it $I_1 = I_1(I_2, h)$, and consequently ω_i are functions of variable I_2. If the ratio of frequencies ω_2/ω_1 depends on I_2 (i.e. is not reducible to a constant), the investigated system is isoenergetically nondegenerate. Then according to /2,3/ there is stability of the variables action. This means that for fairly small ε in the system with Hamiltonian (3.2) the variables I_1, I_2 perpetually remain close to their initial values.

The condition of nondegeneracy is in this problem satisfied.

To check the fulfillment of the condition of nondegeneracy, we consider the identity $H_0(I_1(I_2, h), I_2) = h$. Differentiating it with respect to I_2, as in /8/, we obtain that

$$\omega_2/\omega_1 = -\partial I_1/\partial I_2 \tag{3.3}$$

From the integral (3.1) we find

$$\frac{\partial I_1}{\partial I_2} = \frac{1}{2\pi}\oint\sqrt{A + m\rho^2}\left[\left(\frac{1}{A} - \frac{1}{C}\right)I_2 + \frac{p_\psi\cos\theta - I_2}{A\sin^2\theta}\right] \times \left[2H_0 - \frac{I_2^2}{C} - \frac{(p_\psi - I_2\cos\theta)^2}{A\sin^2\theta} - 2mgf(\theta)\right]^{-1/2} d\theta \tag{3.4}$$

Taking into account (1.2) and (1.3), we obtain from (3.3) and (3.4) that at the isoenergetic level the ratio of frequencies is given by formula

$$\omega_2/\omega_1 = \Delta\varphi/2\pi \tag{3.5}$$

where $\Delta\varphi$ is the angle by which the body turns about the axis of symmetry in time equal to the period of oscillations of angle θ in unperturbed motion (Fig.3,a)

$$\Delta\varphi = \left(\frac{1}{C} - \frac{1}{A}\right)I_2\tau + \int_{\theta_1}^{\theta_2}\frac{I_2 - p_\psi\cos\theta}{A\sin^2\theta}\sqrt{\frac{2(A + m\rho^2)}{h - \Pi_*}}\, d\theta \tag{3.6}$$

Let us show that angle $\Delta\varphi$ depends on I_2. To do this, we investigate the behavior of $\Delta\varphi$ as $I_2 \to \infty$, i.e. when at the initial instant of time the body is rapidy rotating about the axis of symmetry.

Let at $t = 0$, $\psi^\cdot = \theta^\cdot = 0$, $p_\varphi = I_2$, $\theta = \theta_1$. For such initial data $p_\psi = I_2\cos\theta$, $h = mgf(\theta_1) + I_2^2/(2C)$. The equation $h - \Pi_*(\theta) = 0$ is of the form

$$I_2^2\frac{(\cos\theta_1 - \cos\theta)^2}{2A\sin^2\theta} + mgf(\theta) = mgf(\theta_1) \tag{3.7}$$

The angle θ during the motion of the body varies between θ_1 and θ_2, where θ_2 is the root of Eq.(3.7) nearest to θ_1. It can be represented in terms of series in negative powers of I_2

$$\theta_2 = \theta_1 + \frac{2Amg\rho_1}{I_2^2} + \frac{(2Amg\rho_1)^2(2\operatorname{ctg}\theta_1 + \rho_1'/\rho_1)}{2I_2^4} + O\left(\frac{1}{I_2^6}\right) \tag{3.8}$$

where ρ_1 and ρ_1' denote the values of function $\rho(\theta)$ and its derivative when $\theta = \theta_1$. Formula (3.8) makes more accurate the estimate of θ_2 adduced in /1/.

Using (3.8), from (2.3) we obtain the expansion for the period of oscillation of angle θ, and then the expression for the quantity $\Delta\varphi$, determined by the equality (3.6)

$$\Delta\varphi = \frac{14A}{3C}\sqrt{1+\frac{m\rho_1^2}{A}}\left\{1+\frac{13Cmg\rho_1}{35I_2^2}\left[\left(3\frac{A}{C}-2\right)\operatorname{ctg}\theta_1 + \frac{A\rho_1'(A+3m\rho_1^2)}{C\rho_1(A+m\rho_1^2)}\right]\right\} + O\left(\frac{1}{I_2^4}\right) \tag{3.9}$$

For any values of A and C and for any form of surface bounding the body it is always possible by selecting the arbitrary angle θ_1 to obtain that the expression in brackets in (3.9) does not vanish. (An exotic case of $\rho(A+m\rho^2)\sin^{(3-2C/A)}\theta = \text{const}$ is excluded). Thus $\Delta\varphi$ is not reduced to constant value, but depends on I_2, i.e. the condition of nondegeneracy is satisfied.

According to Kolmogorov's theorem /2,3/ the variables "action" are stable for small perturbations of the Hamiltonian H_0. From this immediately follows that the projection of the body momentum on the Gz axis for all t is close to the initial value at $t = 0$. For $0 < \varepsilon \ll 1$ the phase pattern in plane $\theta, \theta^{\cdot}$ only slightly differs from the respective phase pattern of the unperturbed problem. In particular, the range of variation of the angle of nutation varies only slightly, and, also, the character and disposition of traces of the contact point P on the plane and on the surfade bounding the solid body.

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